# An Equitable Solution to the Stable Marriage Problem

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# Presentation Overview

# Introduction

The *stable marriage problem* (SMP) is a typical matching problem.

Input:

- $n$ members of group A (men)
- $n$ members of group B (women)
- each member retains a preference list for each member of the opposite group

**Objective:** identify a *perfect* and *stable* matching between A and B.

Perfect: all members of A and B should be paired with exactly one member of the opposite group.

Stable: there exists no pair of couples $(a_k, b_l)$, $(a_m, b_n)$ such that:

- $a_k$ prefers $b_n$ to $b_l$
- $b_n$ prefers $a_k$ to $a_m$

simultaneously.

# Introduction

**Solution:** Gale-Shapley algorithm

- one group is assigned as the **proposers**, the other is the **acceptors**
- the proposers issue proposals to the other group
- the acceptors evaluate their proposals and marry their most preferable choice
- guarantee for a **stable** solution
- termination in $O(n^2)$ steps

**Inequality:** the proposers are more satisfied than the acceptors

- various real world use cases demand equal treatment to both sides
  - e.g. college admission problem

# Introduction

*Equitable Stable Marriage Problem* (ESMP):

- **perfect** and **stable** matching between A and B
- members from both groups should be equally satisfied

**ESMP:** sex equality minimization is **NP-hard**

**Propose:** *ESMA*, a heuristic algorithm to identify **equal** and **stable** solutions.

# Background

## Gale-Shapley algorithm

Preparation step:

- the input groups are divided to *proposers* and *acceptors*

Step *i*:

- the *proposers*:
  - if married, they do nothing
  - if not married, they propose to their next most preferred choice
- the *acceptors*:
  - receive the proposals
  - engage to the proposer they prefer the most

Termination condition

- everybody is married

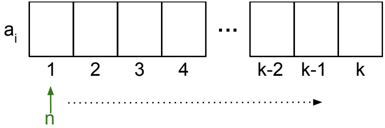
# Background

## Gale-Shapley algorithm: Preference Lists

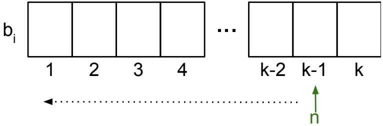
Preference lists:

- ordered lists indicating the preference ordering of each agent for the opposite group members
- index *n* indicating the current fiance OR the next most desirable member

Proposer




Acceptor



# Background

## Gale-Shapley algorithm: Monotonicity

Preference pointers always move to the same direction:

**proposers** pointer starts from the first preference and moves downwards (less satisfied)

**acceptor** pointer starts from a random position and moves upwards (more satisfied)

This monotonicity ensures that:

- the algorithm terminates
- the matching is stable

However:

- the proposers start from higher preference ranks and they are, eventually, favored

# Background
## Swing

**Swing:** an equitable algorithm

Both groups act as proposers and acceptors:

- group A proposes at even steps
- group B proposes at odd steps

Each proposer issues multiple proposals:

- from their first up to their $n^{th}$ choice

Results:

- **stability** is guaranteed
- **termination** is not

Complexity:

- $O(n^2)$ per iteration

# Equitable Stable Marriage Algorithm

## Introduction

Main idea: both groups act as *proposers* and *acceptors* in different steps. This way:

- both groups try to achieve the best for themselves
- the frequency of picking a group as proposers impacts the group's final satisfaction

However:

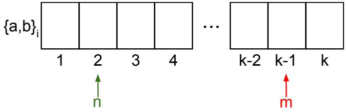
- complex indexing needed for the preference list
- the *monotonicity* is lost
- the GS termination condition (everyone married) is not enough for stability

# Equitable Stable Marriage Algorithm

## Preference Lists and Members states

Two indices for each agent:

- $n$, indicating the next most preferable choice
- $m$, indicating the preference rank of the current fiance (if any)




Agent states:

| **Status** | single | motivated | content |
|---|---|---|---|
| **index** | $m = \infty$ | $m > n$ | $m = n$ |

single/motivated the agent proposes to their next preference

content the agent stops

If an agent with rank $r < n$ proposes, them $m = n = r$ ($n$ and $m$ move upwards).

# Equitable Stable Marriage Algorithm

## Propose and evaluate functions

Function **EVALUATE**$\{a, b\}$
$old = M(a)$
**if** $a.m == \infty$ or $pr_a[b] < a.m$ **then**
    $old.m = \infty$
    $a.m = pr_a[b]$
    **if** $a.n > pr_a[b]$ **then**
        $a.n \leftarrow pr_a[b] + 1$
**else**
    **return** false
**return** true

$pr_a[b]$: the preference rank of agent $a$ for agent $b$

$\ell_a[a.n]$: the agent that $a$ prefers $n^{th}$

Function **PROPOSE**$\{a\}$
$old = M(a)$
**if** $a.n < a.m$ **then**
    $b = \ell_a[a.n]$
    **if** EVALUATE$(b, a)$ **then**
        $old.m = \infty$
        $a.m = a.n$
    **else**
        $a.n = a.n + 1$

# Equitable Stable Marriage Algorithm

## Algorithm overview & stability

**Require:** $A, B$
**Ensure:** A stable matching $w$

$w = \emptyset$
$k = 0$
**while** not (everyone is content) **do**
    $k += 1$
    $P =$ PICK_PROPOSERS$(A, B, k)$
    **for all** $p \in P$ **do**
        PROPOSE$(p)$
**for all** $a \in A$ **do**
    $w = (a, M(a))$
**return** $w$

**Stability theorem**: if the Algorithm terminates, it finds a stable matching.

# Equitable Stable Marriage Algorithm

## Circular dependencies

When an agent receives a proposal from a more preferable choice than $n$, $m$ and $n$ are moved to previous positions of the preference list

- this creates *circularity*
- circularity may lead to infinite loops: a set of agents may re-iterate their preference lists endlessly

Observation:

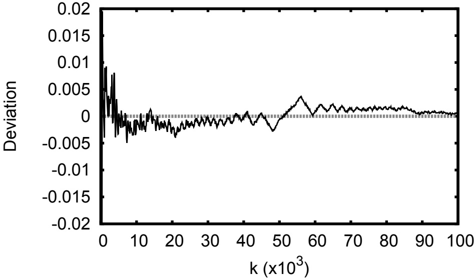
- the pattern of picking the proposer group, affects the appearance of endless algorithm loops
- the pattern should be:
  - state agnostic but reproducible
  - aperiodic but fair

# Equitable Stable Marriage Algorithm

## Assigning Proposers

Function **PICK_PROPOSERS**$\{A, B, k\}$
**if** $\sin(k^2) \geq 0$ **then**
    **return** $A$
**else**
    **return** $B$

"Fair" function between the opposite groups:



# Equitable Stable Marriage Algorithm

## Performance Optimization

No need to reiterate over the entire preference list:

- when *content*, the agent keeps track of the proposals that he denies
- when *single* or *motivated*, the agent only re-proposes to agents that:
  - they have never received a proposal from him
  - they have proposed to him while content

Not a stability breach:

- absence of proposals means absence of interest (from the other agent)

The optimization is easy to implement and speeds up the algorithm execution.

# Evaluation

## Performance metrics

Egalitarian cost:

$$c(M) = \sum_{(m,w)\in M} pr_m(w) + \sum_{(m,w)\in M} pr_w(m)$$

Sex equality cost:

$$d(M) = \left| \sum_{(m,w)\in M} pr_m(w) - \sum_{(m,w)\in M} pr_w(m) \right|$$

$pr_a[b]$: the preference rank of agent $a$ for agent $b$

# Evaluation

## Data

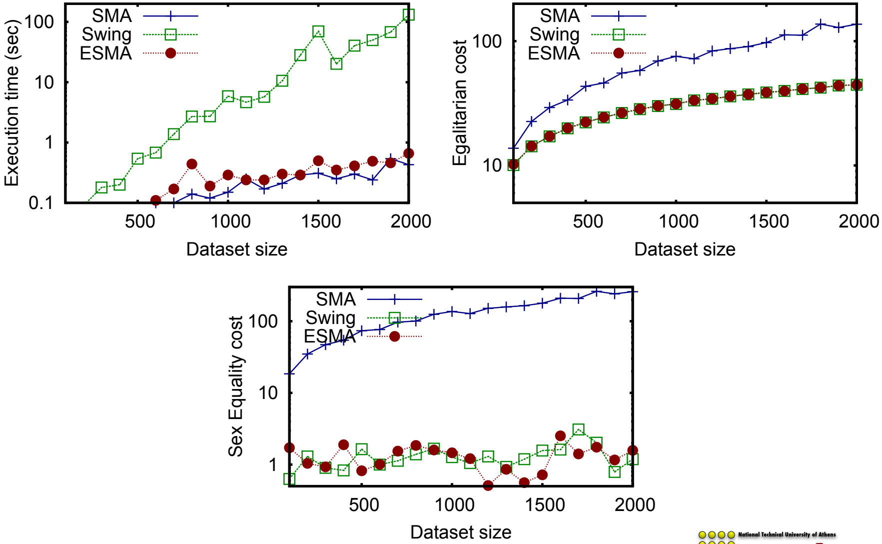
Synthetics datasets following different distributions:

- Uniform: random assignment of scores to different agents
- Gaussian: default order of preference lists and adding *Gaussian noise* with different amplitude and resorting
- Discrete Regions: divide each preference list in two regions (*Hot* and *Cold* region) and uniform distribution within a region

- 100 upto 2000 agents
- 5 variants per size
- Comparing **ESMA** to **Gale-Shapley** and **Swing**
- Each dataset that leads **Swing** to termination is tested

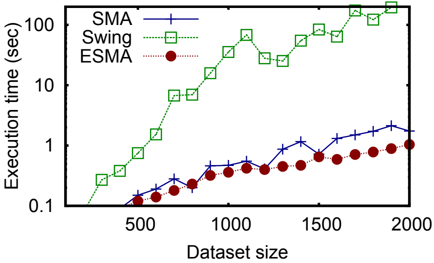
# Evaluation

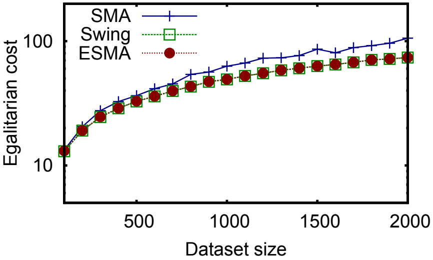
## Results vs Dataset size - Uniform distribution

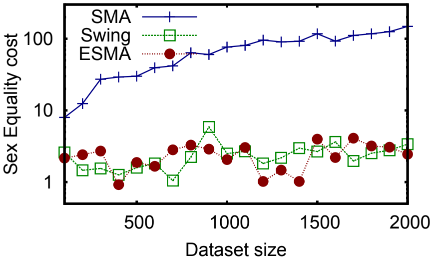


# Evaluation

## Results vs Dataset size - Gaussian distribution (noise 20%)

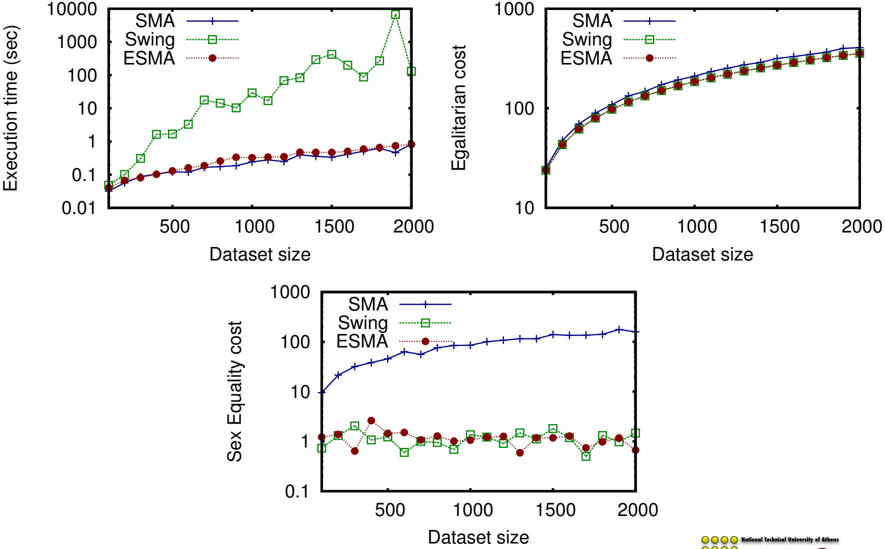


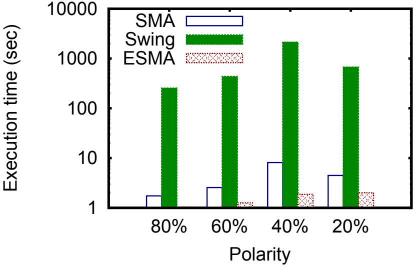
# Evaluation

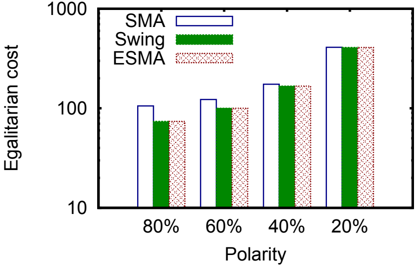
## Results vs Dataset size - Discrete Regions (hot region: 20%)

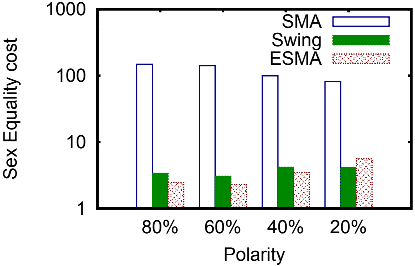


# Evaluation

## Results vs data polarity - Gaussian distribution



# Conclusions

In this paper:

- revisited the *Equitable Stable Marriage Problem*
- observed that the lack of *monotonicity* introduces problems with stability or termination
- identified that a non periodic pattern of assigning proposers solves the problem of termination
- proposed *ESMA*, that:
  - provides "fair" solutions in time equivalent to the non-equitable Gale-Shapley algorithm
  - works satisfactorily for all the tested data distributions

# Thank you!

Thank you! Questions?